Journal of Statistical Physics, Vol. 66, Nos. 3/4, 1992

A Mean-Field Limit for a Class of Queueing Networks

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Received February 13, 1991; final August 22, 1991

A model of centralized symmetric message-switched networks is considered, where the messages having a common address must be served in the central node in the order which corresponds to their epochs of arrival to the network. The limit $N \to \infty$ is discussed, where N is the branching number of the network graph. This procedure is inspired by an analogy with statistical mechanics (the mean-field approximation). The corresponding limit theorems are established and the limiting probability distribution for the network response time is obtained. Properties of this distribution are discussed in terms of an associated boundary problem.

KEY WORDS: Message-switched network; synchronization constraint discipline (FEFS); starlike configuration graph; infinite branching limit; mean-field or Poisson approximation; generalized Lindley equation and its stationary solution; associated boundary problem.

1. INTRODUCTION

One of the most popular tools in the theory of switching networks used for evaluating or estimating various network characteristics is the so-called

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0022-4715/92/0200-0803$06.50/0 © 1992 Plenum Publishing Corporation

Poisson (or independence) approximation. Physically speaking, this means that in a given network, all the flows (of customers, messages, calls, packets, programs, etc., depending on the application domain) are set to be Poissonian and, in an appropriate sense, are independent for different servers. Such an approximation was proposed by Kleinrock[15] for centralized starlike message-switched networks with the queueing discipline FCFS, first come—first served (with respect to the arrival epochs of messages in the subsequent nodes of their paths). The problem of rigorously proving this conjecture for that type of network was investigated by Dobrushin and Sukhov[11] and Brown and Pollett.[9] For other network classes (circuit-switched, packet-switched, hybrid) the Poisson approximation was studied rigorously in a series of papers by R. L. Dobrushin, M. Ya. Kelbert, A. N. Rybko, A. L. Stoljar, and Yu. M. Sukhov (we refer the reader to the review paper of Kelbert and Sukhov[14] for the references). These investigations were inspired essentially by a fruitful analogy between queueing network theory and statistical mechanics. Such an analogy may be established by considering the "size" parameters (the number of nodes or communication lines) as "extensive" characteristics and "the single-node" parameters (intensity rates of exogenous flows, probability distributions related to a single customer) as "intensive" characteristics of a network. Various limiting procedures are then available, where extensive parameters grow to infinity and intensive ones remain fixed (or are related to extensive parameters in some other specific way). The Poisson approximation is associated thereby with the well-known mean-field approximation in statistical mechanics. The limiting mean-field picture is characterized by the fact that each single customer is "plugged" into an independent environment which is generated by other customers of the same type. Physically speaking, a maximal independence is attained, within "reasonable" limits provided by the network structure.

In this paper we discuss the Poisson approximation for a class of message-switched networks which was introduced into the mathematical literature (in a more general form) by F. Baccelli *et al.* (see the review paper of Baccelli and Makowski[3] and the references therein, and Baccelli and Liu[4]). The main feature of this type of network is the so-called synchronization constraint rule. In a simple form, this condition requires that for any pair of messages which have intersecting paths, the message which arrived earlier in the network (i.e., at the initial node of its path) must be served earlier in any node which is common to both paths. One result concerning synchronization constraint networks is that such a network has a nonzero capacity region.[5]

We consider in this paper a two-stage feedforward queueing model with N single-server queues at the first stage, each with its own Poisson input flow, and N single-server queues at the second stage. Each arriving message has two jobs to complete, one at each stage, requiring service times (l_0, l_1); the sequence of these vectors can be thought of as being handed out independently upon arrival according to a fixed distribution F. Each of the N first-stage queues defines a standard FCFS $M/GI/1/\infty$ queueing system (the one with Poisson input flow and i.i.d. service times). These N systems are entirely independent and all have the same arrival rate and the same service (transmission) time distribution. Each message, upon arrival at the first stage, randomly selects (with equal probability and independently of the past) one of the N queues to attend at the second stage. Order of service at any of the second-stage queues is determined by the following rule: messages must be served at a queue in the order in which they arrived *exogenously to the first stage.* In particular, this discipline is not work-conserving, because a waiting message cannot begin service until its predecessor has been served (and this predecessor may not be at the second stage yet). Equations (1.2) and (1.3) below describe the evolution of message waiting times.

The result of the paper may be formulated so that in the limit as N tends to infinity, the message's stationary waiting times satisfy Eq. (1.6) (see below). Moreover, messages from any fixed Poisson exogenous flow have mutually conditionally independent second-stage waiting times given all their arrival epochs, first-stage waiting times, and service time vectors. Similarly, messages from a (Poisson) flow corresponding to a fixed second-stage queue have mutually conditionally independent first-stage waiting times given all their arrival epochs, second-stage waiting times, and service time vectors. Intuitively, this is based upon the fact that any two messages at a second-stage queue will have come from the same first-stage queue with negligible probability. In addition, when N is large, the arrival process to a fixed second-stage queue is approximately Poisson, since it is simply the superposition of many independent $M/GI/1/\infty$ departure processes, each one contributing a small proportion. Hence the limiting process is Poisson and with i.i.d. marks which are formed by the service time vectors and first-stage waiting times. The service on the second stage generates, in the $N \to \infty$ limit, nothing but the standard $M/GI/1/\infty$ queue. This leads to the picture outlined before.

Let us now pass to a detailed description in a network terminology. We consider a starlike network which consists of a central node M and of a collection of peripheral sources $C_1, ..., C_N$ and peripheral addresses $C'_1, ..., C'_N$ (see Fig. 1). Each oriented edge of the graph represents a communication line (server) which transmits the messages in the corresponding direction. It is convenient to place the server at the input of each line (see Fig. 2). We shall denote by a_j the input line leading from C_j to M, as well

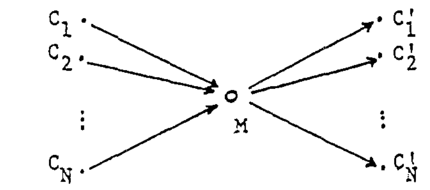

Fig. 1

as the corresponding server, and by a'_j the output line leading from M to C'_j as well as the corresponding server, $1 \leqslant j \leqslant N$.

Suppose that the input of any peripheral server a_j is given as an external flow ξ_j which is composed of exogenous messages. Every message from ξ_j is identified with a triple $y = [t; (L, d)]$, where $t = t(y)$ is the epoch of arrival into the network, $L = L(y)$ is the length vector (l_0, l_1) with components $l_0 = l_0(y)$ and $l_1 = l_1(y)$, and $d = d(y) \in \{1,..., N\}$ is the address. We can say that ξ_j is a random marked point process (rmpp) with marks $(L, d) \in \mathbb{R}_+^2 \times \{1,..., N\}$. We suppose that the rmpp's ξ_j, $1 \leqslant j \leqslant N$, are i.i.d. Furthermore, we assume that ξ_j is Poisson of intensity $\lambda > 0$ and with (conditionally) i.i.d. marks (L, d). We also assume that the marginal distribution of the mark value is given by a fixed probability measure $\bar{G}$ on $\mathbb{R}_+^2 \times \{1,..., N\}$, which is the product $F \times G$, where F is a probability measure on $\mathbb{R}_+^2$ with marginals F_0 and F_1, and G is the equiprobability distribution on $\{1,..., N\}$. Throughout the paper the measures F_0 and F_1 are supposed to satisfy the following conditions:

$$E_{F_0} \exp(bl_0) < \infty \quad \text{for some} \quad b > 0, \qquad E_{F_1} l_1 < \infty \tag{1.1}$$

(E_P denotes the expectation with respect to a probability measure P).

For the basic notions and facts of the rmpp theory and its relations to queueing theory see, e.g., the book by Franken *et al.*,[13] the monograph by

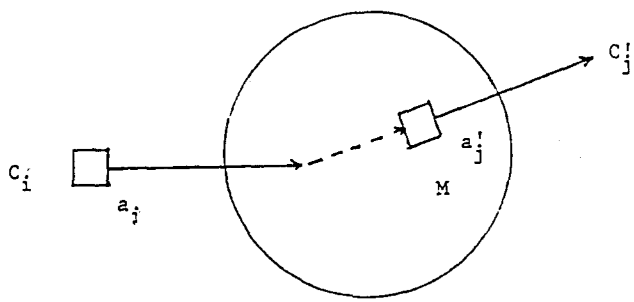

Fig. 2

Baccelli and Brémaud,[1] or the aforementioned review paper of Kelbert and Sukhov.[14]

The collection of rmpp's $\{\xi_j, 1 \leqslant j \leqslant N\}$ will be called the external family.

The network transmission (service) rule is as follows. At any given time, each line a_j, a'_j, $1 \leqslant j \leqslant N$, can be used for transmitting at most one message and no preemption is allowed. A message $y = [t; (L, d)]$ which was initiated in a source C_j, i.e., in the rmpp ξ_j, must subsequently be transmitted along the lines a_j and a'_d (the message-switching principle). Correspondingly, the network response time of the message y is divided into two parts. In the first stage, the message is waiting for the line a_j and is then transmitted during the time l_0. The queue at the server a_j is formed by the rmpp ξ_j and governed by the usual FCFS discipline. By convention, we shall call this queue the input queue. Under our assumptions the input queue is merely the $M/GI/1/\infty$ queue with arrival intensity λ and service time distribution F_0.

Immediately after completing the transmission along the line a_j, the message starts its second stage. This stage consists in waiting for the line a'_d and then in transmitting along this line during the time l_1. The queue for the server a'_d is called the output queue. It consists of the messages of the various rmpp's ξ_i, $1 \leqslant i \leqslant N$, with the same address. The output queue is governed by a specific discipline which is denoted by FEFS [first emitted (in the corresponding external rmpp ξ_i)—first seved]. This means that the message y must wait until all the messages with the same address d, which have been emitted in their sources before the epoch t, finish their transmission along the line a'_d.

Equivalently, one can think that at the moment of its arrival to the network a message starts waiting in both queues, input and output. But the service on the corresponding line a'_d may only start after completing the transmission along the line a_j.

After completing the transmission along the line a'_d, the message y leaves the network.

We are interested in studying the waiting and response time distributions for a single message. The formal procedure consists in introducing the corresponding random variables and in constructing a family of rmpp's $\{\eta_j\}$ which provides a stationary solution to a system of equations associated with such a network. We have in mind to add a new component to the mark (L, d) of an external message, characterizing the process of transmission of the message along its path (a_j, a'_d). It is convenient to take, as a new component, the vector $W = (w_0, w_1) \in \mathbb{R}^2_+$ with entries $w_0, w_1 \geqslant 0$, which are the waiting times for beginning the transmission along the lines a_j and a'_d, respectively (all the waiting times are counted from the epoch t

of the arrival to the network). The network response time for our message will be $T_1 = w_1 + l_1$, whereas $T_0 = w_0 + l_0$ will be the sojourn time in the peripheral source. Of course, the values w_0, w_1 cannot be arbitrary: they will be coupled by the system of equations.

The precise formulation of the problem is as follows. We want to construct the joint probability distribution of a family of rmpp's η_j, $1 \leqslant j \leqslant N$, with marks $(L, d, W) \in \mathbb{R}_+^2 \times \{1,..., N\} \times \mathbb{R}_+^2$ satisfying the two conditions:

I. The projection $(L, d, W) \to (L, d)$ transforms the joint distribution of the family $\{\eta_j\}$ into the joint distribution of the family $\{\xi_j\}$.

II. With probability one, the marks (L, d, W) and the arrival epochs t obey the system of equations which is written below.

We now formulate our system of equations. Let us start with some technical remarks. In the main part of the paper, we use an approach based on the rmpp theory. In particularly, when speaking of a realization of a rmpp with marks in a (standard Borel) space K, we have in mind a σ-finite integer-valued measure on $\mathbb{R} \times K$, where all the points enter with multiplicity one. Such a property of all the rmpp's under consideration is guaranteed with probability one by our assumptions about the family $\{\xi_j\}$.

Given a family of realizations $\bar{\omega}_j$, $1 \leqslant j \leqslant N$, of rmpp's with marks in $\mathbb{R}_+^2 \times \{1,..., N\} \times \mathbb{R}_+^2$ and a message $\bar{y} = [t; (L, d, W)]$ belonging to $\bar{\omega}_i$, we say that a message $\bar{y}'$ of $\bar{\omega}_i$ (respectively, a message $\bar{y}''$ with $d(\bar{y}'') = d(\bar{y}) = d$, of any of $\bar{\omega}_k$, $1 \leqslant k \leqslant N$) *precedes* the message $\bar{y}$ on the input line a_i (resp. on the output line a'_d) if:

(i) $t(\bar{y}') < t(\bar{y})$ [resp. $t(\bar{y}'') < t(\bar{y})$].

(ii) There is no message $\bar{y}^*$ in $\bar{\omega}_i$ with $t(\bar{y}') < t(\bar{y}^*) < t(\bar{y})$ [resp. there is no message $\bar{y}^*$ in $\bar{\omega}_k$, $1 \leqslant k \leqslant N$, with $t(\bar{y}'') < t(\bar{y}^*) < t(\bar{y})$ and $d(\bar{y}^*) = d$].

We consider a family of stationary rmpp's $\{\eta_j\}$ such that with probability one, for any $j = 1,..., N$ and for every message $\bar{y}$ from $\bar{\omega}_j$, there exists just one message $\bar{y}'$ which precedes $\bar{y}$ on the line a_j and just one message $\bar{y}''$ which precedes $\bar{y}$ on the line a'_d, $d = d(\bar{y})$, and such that the entries of the waiting time vectors $W(\bar{y})$, $W(\bar{y}')$, and $W(\bar{y}'')$ for the messages $\bar{y}$, $\bar{y}'$, and $\bar{y}''$, respectively, are related by the system of equations

$$w_0(\bar{y}) = \max[0, w_0(\bar{y}') + l_0(\bar{y}') - (t(\bar{y}) - t(\bar{y}'))] \tag{1.2}$$

and

$$w_1(\bar{y}) = \max[w_0(\bar{y}) + l_0(\bar{y}), w_1(\bar{y}'') + l_1(\bar{y}'') - (t(\bar{y}) - t(\bar{y}''))] \tag{1.3}$$

We shall say that such a family $\{\eta_j\}$ provides a stationary solution to the system of equations associated with the external family $\{\xi_j\}$.

Equations (1.2) and (1.3) translate the synchronization constraint described before. Namely, (1.2) is nothing more than the $M/GI/1/\infty$ equation for the actual waiting time in an input queue, whereas (1.3) indicates how the output queue is attended. In order to establish (1.3), one must keep in mind that the message $\bar{y}$ will start its service on the line a'_d, $d = d(\bar{y})$, precisely at time

$$\max[t(\bar{y}) + w_0(\bar{y}) + l_0(\bar{y}), t(\bar{y}'') + w_1(\bar{y}'') + l_1(\bar{y}'')]$$

A similar problem may be stated for the cutoff external family $\{\xi_{j,t^0}\}$, which is composed of the restrictions $\xi_{j,t^0} = \xi_j 1_{[t^0,\infty)}$ of the rmpp's ξ_j to the half-axis $[t^0, \infty)$. Let some positive initial workload z^0 be given, describing the initial condition. To any initial condition of this type and to any realization of the cutoff external family $\{\xi_{j,t^0}\}$, one can associate a unique transient solution of the system of equations (1.2), (1.3),[3] which will be denoted by $\{\eta_{j,t^0,z^0}\}$.

To guarantee the existence and uniqueness of a stationary regime in the network under consideration, we need the nonoverload condition given by

$$\lambda\mu^{-1} < 1 \tag{1.4}$$

where

$$\mu^{-1} = \max(\mu_0^{-1}, \mu_1^{-1}), \qquad \mu_0^{-1} = E_{F_0} l_0, \qquad \mu_1^{-1} = E_{F_1} l_1$$

Notice that the bound (1.4) does not depend on N.

From general results given in Baccelli *et al.*,[2-4] one can deduce the following assertion.

Theorem 1. Let the nonoverload condition (1.4) hold. Then there exists a unique stationary family $\{\eta_j\}$ which provides the unique stationary solution of the system of equations (1.2)–(1.3) associated with the external family $\{\xi_j\}$.

Moreover, for any initial condition z^0

$$\eta_j = \lim_{t^0 \to -\infty} \eta_{j,t^0,z^0} \tag{1.5}$$

The convergence of rmpp's means here the usual weak convergence (convergence in distribution).

Remark. In fact, we have a stronger convergence where the projections of rmpp's to any bounded time interval converge in total variation.

Our goal in this paper is to study the asymptotic behavior of the distribution of the family $\{\eta_j\}$ when $N \to \infty$. The images of the rmpp's ξ_j and η_j under the projections $(L, d) \to L$ and $(L, d, W) \to (L, W)$ are denoted by $\tilde{\xi}_j$ and $\tilde{\eta}_j$, respectively.

Before stating the main theorems, we introduce some definitions that will be repeatedly used throughout the paper. We consider the stochastic equation

$$w \cong \max(T, w + l - u) \tag{1.6}$$

where the symbol $\cong$ stands for equality in law, and where the random variables w, l, T, and u are nonnegative and mutually independent. The unknown is the distribution function of w, and the following statistics are known: l is distributed according to F_1, u is exponentially distributed with parameter λ, and T is distributed like the stationary sojourn time in an $M/GI/1/\infty$ queue with parameter λ, F_0. We denote the distribution of the sojourn times T by $\Psi_0 = \Psi_0(\lambda, F_0)$, and that of the waiting time by $\Psi = \Psi(\lambda, F_0)$.

As we will see later, under the stability condition (1.4), Eq. (1.6) has a unique solution, which will be denoted by $\Phi = \Phi(\lambda, F_0, F_1)$.

We remind the reader that the Palm distribution of a given rmpp η describes a sort of conditional distribution generated by η under the "condition" that, at a given time, a message appears in this rmpp.

Theorem 2. Assume that condition (1.4) is fulfilled. Given any finite set $J \subset \mathbb{N}$, the rmpp's $\tilde{\eta}_j = \tilde{\eta}_j^{(N)}$, $j \in J$, converge, as $N \to \infty$, to limiting rmpp's $\tilde{\eta}_j^{(\infty)}$, $j \in J$, which are i.i.d. The marginal distribution of a single rmpp $\tilde{\eta}_j^{(\infty)}$ has the following properties:

(a) The projection $(L, W) \to L$ transforms the rmpp $\tilde{\eta}_j^{(\infty)}$ into the rmpp $\tilde{\xi}_j$.

(b) The components w_0 are solutions of Eqs. (1.2), and the Palm distribution of w_0 (wrt the rmpp $\tilde{\eta}_j^{(\infty)}$) coincides with Ψ.

(c) The Palm distribution of w_1 is given by Φ.

Remark. The properties (a)–(c) do not determine completely the distribution of the rmpp $\tilde{\eta}_j^{(\infty)}$. In order to do so, one must specify the correlation between w_1, the epochs of the point process, and the other components of the marks.

Let $\mathcal{M}$ be the σ-algebra generated by the epochs of $\tilde{\eta}_j^{(\infty)}$ and by the sequence of random variables $L(\bar{y})$ and $w_0(\bar{y})$. The following characterization of the conditional distribution of the sequence of variable $w_1(\bar{y})$ given $\mathcal{M}$ holds true:

(d) For different messages of $(\bar{y}_1,..., \bar{y}_n)$ of $\tilde{\eta}_j^{(\infty)}$, the variables $w_1(\bar{y}_i)$, $1 \leqslant i \leqslant n$, are mutually conditionally independent, given $\mathcal{M}$.

(e) The conditional distribution of $w_1(\bar{y})$ given $\mathcal{M}$ is

$$\max(w_0(\bar{y}) + l_0(\bar{y}), w + l - u)$$

where w, l, and u are mutually independent, and w has for distribution Φ, l has for distribution F_1, and u is exponentially distributed with parameter λ.

It is useful to compare Theorem 2 with a "dual" assertion concerning the rmpp η'_k which is obtained when superposing the rmpp's $\{\eta_j\}$, $j = 1,..., N$, and when extracting the points $\bar{y}$ with $d(\bar{y}) = k$ and omitting the value $d(\bar{y})$ of the mark [i.e., by considering the reduced mark (L, W)].

Theorem 3. Under the condition (1.4), for any fixed finite set $J' \subset \mathbb{N}$ (or a collection of lines a'_k, $k \in J'$) the rmpp's $\eta_k'^{(N)}$, $k \in J'$, converge, as $N \to \infty$, to the limiting i.i.d. rmpp's $\eta_k'^{(\infty)}$, $k \in J'$. The marginal distribution of a single rmpp $\eta_k'^{(\infty)}$ has the following properties:

(a) The projection $(L, W) \to L$ leads to the rmpp $\tilde{\xi}_k$.

(b) For different messages of the rmpp $\eta_k'^{(\infty)}$, the marks w_0 are conditionally i.i.d., given the σ-algebra generated by the epochs of $\eta_k'^{(\infty)}$, and with Palm distribution Ψ.

(c) The Palm distribution of w_1 in the rmpp $\eta_k'^{(\infty)}$ coincides with that in the rmpp $\tilde{\eta}_j^{(\infty)}$.

Remark. Again these properties do not determine the distribution of the rmpp. The conditional distribution of the sequence $w_1(\bar{y})$ given $\mathcal{M}$ is deterministic and given by Eq. (1.3).

Notice that the distribution Ψ that shows up in Theorems 2 and 3 may be written in a simple explicit form using the Pollaceck–Khinchin formulas. The distribution of Φ can also be evaluated, in a more complicated way, for a particular case of exponential distributions for u and T (and even in the case where the distribution law for T has a rational Laplace–Stieltjes transform).[2] In Theorem 4 we give another characterization of Φ in the case where l and u have exponential distributions, and T has a general distribution.

First, we replace (1.6) by the stochastic equation for the variable $X \cong w + l - u$:

$$X \cong l - u + \max(X, T) \tag{1.7}$$

As before, we can assume that the variables X, T, l, and u in (1.7) are independent. Moreover, it is assumed that both l and u are exponentially distributed with expectations μ^{-1} and λ^{-1}, respectively.

Furthermore, we rewrite (1.7) in terms of the distribution functions:

$$\mathscr{F} = g * (\mathscr{F}\mathscr{F}_0) \tag{1.8}$$

Here $\mathscr{F}$ is the distribution function of the variable X, g is the probability density of the difference $l-u$, $\mathscr{F}_0$ is the distribution function of the variable T, and the asterisk stands for convolution. Notice that (1.8) is a linear equation; we are interested in monotone nondecreasing solutions $\mathscr{F}$ satisfying the conditions

$$\mathscr{F}(-\infty) = 1 - \mathscr{F}(\infty) = 0 \tag{1.9}$$

Theorem 4. Let $\lambda < \mu$, $\mathscr{F}_0(x) = 0$, for $x < 0$, and $\int dx\,[1 - \mathscr{F}_0(x)] < \infty$. Then the (unique) bounded nondecreasing solution (1.8), (1.9) has the following properties:

1. $\mathscr{F}$ is of class C^1.

2. $\mathscr{F}$ has one-side second derivatives $\mathscr{F}''_-$ and $\mathscr{F}''_+$ which satisfy the equations

$$\mathscr{F}''_-(x) + a\mathscr{F}'(x) = \lambda\mu\mathscr{F}(x)[1 - \mathscr{F}_0(x-0)] \tag{1.10}$$

$$\mathscr{F}''_+(x) + a\mathscr{F}'(x) = \lambda\mu\mathscr{F}(x)[1 - \mathscr{F}_0(x+0)] \tag{1.11}$$

where $a = \mu - \lambda$.

3. If x is a point of continuity for $\mathscr{F}_0$, then $\mathscr{F}''(x)$ exists and satisfies the common equations (1.10)–(1.11).

4. For $x < 0$

$$\mathscr{F}(x) = \mathscr{F}(0) \exp \lambda x \tag{1.12}$$

5. For $x = 0$

$$\mathscr{F}'(0) = \lambda\mathscr{F}(0) \tag{1.13}$$

The rest of the paper is organized as follows. In Section 2 we give a short direct proof of Theorem 1. Though, as we have noted before, the assertion of Theorem 1 may be deduced from general results of Baccelli *et al.*,[2–4] the direct proof is useful because it provides some insight into the arguments used in the proofs of Theorems 2 and 3. The proofs of these theorems are given in Section 3, while Section 4 is devoted to the proof of Theorem 4. Some further properties of the boundary problem (1.10)–(1.13) are also discussed there.

2. EXISTENCE AND UNIQUENESS OF A STATIONARY REGIME

Since the distribution Ψ of w_0 is well known, we shall concentrate on the distribution Φ of w_1. It is convenient to consider a slightly more general situation than in Theorem 1 and to give an alternative formulation of the existence and uniqueness theorem. The notations of this section are independent of those introduced in Section 2.

Let $\{(u_n, v_n), n \in \mathbb{Z}\}$ be a stationary ergodic sequence with values in $\mathbb{R} \times \mathbb{R}_+$. We are interested in stationary solutions of the equation

$$w_n = \max(v_n, w_{n-1} + u_{n-1}) \tag{2.1}$$

We shall call (2.1) the generalized Lindley equation.

Theorem 5. 1. If the sequence (u_n, v_n) satisfies the relation

$$Eu_n < 0, \qquad Ev_n < \infty \tag{2.2}$$

then (a) there exists a unique stationary solution w_n, $n \in \mathbb{Z}$, to (2.1) and (b) for any value $w^0 \geqslant 0$ and any finite set $I \subset \mathbb{Z}$

$$\lim_{n_0 \to -\infty} \operatorname{Var}[\{w_n, n \in I\}, \{w_n(n^0, w^0), n \in I\}] \tag{2.3}$$

where $w_n(n^0, w^0)$, $n \geqslant n^0$, is the solution of (2.1) with initial condition $w_{n^0} = w^0$.

2. Suppose that the values v_n, $n \in \mathbb{Z}$, are i.i.d. and independent of $\{u_n\}$. Suppose in addition that a stationary solution $\{w_n\}$ to (2.1) exists. Then

$$Eu_n \leqslant 0, \qquad Ev_n < \infty$$

If in addition we assume that the random variables u_n, $n \in \mathbb{Z}$, are i.i.d. and not identically 0, then

$$Eu_n < 0 \tag{2.4}$$

Remarks. 1. Stochastic equations of the form (2.1) have been studied by Borovkov,[7] Chapter IV, Section 4, where a result similar to Theorem 5 has been obtained. See also Chapter 3 of ref. 6.

2. In assertion 1(a) of Theorem 5 we have in mind that there exists a strong stationary solution of (2.1) which is unique in the class of weak stationary solutions thereof (more precisely, in the class of weak solutions which are bounded in probability). See Kelbert and Sukhov[14] for details.

3. The result of Theorem 5 may be formulated in terms of coincidence of maximal and minimal solutions to (2.1) (see the papers of Brandt[8]).

To prove Theorem 5, we use the following lemma.

Lemma 2.1. Let $\{u_n\}$ be a stationary ergodic sequence.

(i) If $E|u_n| < \infty$, then $(1/n)\, u_n \to 0$ a.e. when $n \to \infty$.

(ii) If the random variables u_n are i.i.d. and

$$P(\sup_{n \geqslant 1} n^{-1} u_n < \infty) > 0$$

then $Eu_n^+ < \infty$, where $u_n^+ = \max(0, u_n)$.

Proof.[6] Let $n \geqslant 1$. We have

$$\frac{u_n}{n} = \frac{u_0}{n} + \frac{1}{n} \sum_{j=1}^{n} (u_j - u_{j-1})$$

Clearly, $(1/n)\, u_0 \to 0$ a.e. Furthermore, according to the strong law of large numbers, $(1/n) \sum_{j=1}^{n} (u_j - u_{j-1}) \to E(u_1 - u_0)$ a.e. This proves the assertion (i) of the lemma.

If the random variables u_n are i.i.d., then

$$\begin{aligned} P\left(\sup_{n \geqslant 1} \frac{u_n}{n} < N\right) &= \prod_{n=1}^{\infty} P(u_1 < nN) = \prod_{n=1}^{\infty} [1 - P(u_1 \geqslant nN)] \\ &\leqslant \prod_{n=1}^{\infty} \exp(-P(u_1 \geqslant nN)) \end{aligned}$$

But

$$\sum_{n=1}^{\infty} P(u_1 \geqslant nN) \geqslant \int_1^{\infty} dx\, P(u_1 \geqslant xN)$$

and the condition $Eu_1^+ = \int_0^\infty dx\, P(u_1 \geqslant x) = \infty$ implies that

$$P\left(\sup_{n \geqslant 1} \frac{u_n}{n} < N\right) = 0$$

for any $N \geqslant 1$ and hence

$$P\left(\sup_{n > 1} \frac{u_n}{n} < \infty\right) = 0$$

This proves (ii).

Proof of Theorem 5. 1. A natural solution of (2.1) is given by

$$w_n = \max\left[\sup_{r \geqslant 1}\left(v_{n-r} + \sum_{n-r \leqslant m < n} u_m\right), v_n\right] \tag{2.5}$$

The RHS of (2.5) is obtained by passing to the limit $r \to \infty$ in the following expression for $w_n(n-r, w^0)$, the solution of the generalized Lindley equation with the initial condition $w_{n-r} = w^0$:

$$\begin{aligned} w_n(n-r, w^0) = \max(&v_n, v_{n-1} + u_{n-1}, v_{n-2} + u_{n-2} + u_{n-1}, \ldots, \\ &v_{n-r+1} + u_{n-r+1} + \cdots + u_{n-1}, w^0 + u_{n-r} + \cdots + u_{n-1}) \end{aligned} \tag{2.6}$$

Notice that $w_n(n-r, w^0)$ is increasing in w^0 and, for $w^0 = 0$, increasing in r.

Our problem reduces now to checking whether the following assertions hold, under the condition (2.2): (i) With probability one, the values w_n defined in (2.5) are finite for all $n \in \mathbb{Z}$ and, moreover, the supremum is reached for some finite $r \geqslant 1$; and (ii) for any $n \in \mathbb{Z}$ and $w^0 > 0$ the probability that $w_n(n-r, 0) < w_n(n-r, w^0)$ tends to zero as $r \to \infty$.

The assertion (a) follows immediately from the strong law of large numbers for $\{u_n\}$ and assertion (i) of Lemma 2.1 for $\{v_n\}$. To prove (b), compare the RHS of (2.6) for $w^0 > 0$ and $w^0 = 0$. It suffices to check that for any $n \in \mathbb{Z}$ and any $w^0 > 0$

$$\lim_{r \to \infty} P(w^0 + u_{n-r} + \cdots + u_{n-1} > 0) = 0$$

But this follows immediately from the law of large numbers for u_n. This completes the proof of the assertion 1.

2. Notice that the RHS of (2.5), which coincides with $\lim_{r \to \infty} w_n(n-r, 0)$ (provided that this limit is finite), gives a solution of the generalized Lindley equation. Moreover, this is the minimal solution: for every solution $\bar{w}_n$ the bound $w_n \leqslant \bar{w}_n$ holds for all $n \in \mathbb{Z}$. Therefore, the condition of the assertion 2 of Theorem 5 implies that the RHS of (2.5) is finite a.e. Hence, a.e.

$$\sup_r \sum_{n-r<m<n} u_m < \infty$$

which implies that $Eu_n \leqslant 0$ and, if u_n are i.d.d. and not identically 0, that $Eu_n < 0$. Furthermore, the following inequalities hold:

$$
\begin{aligned}
P(w_n < N) &\leqslant P\left(\sup_{r \geqslant 1}\left(v_{n-r} + \sum_{n-r \leqslant m < n} y_n - N\right) < 0\right) \\
&\leqslant P\left(\sup_{r \geqslant N}\left(v_{n-r} + \sum_{n-r \leqslant m < n} u_m - r\right) < 0\right) \\
&\leqslant P\left(\sup_{r \geqslant N}\left(\frac{1}{r} v_{n-r} + \frac{1}{r} \sum_{n-r \leqslant m < n} u_m - 1\right) < 0\right) \\
&\leqslant P\left(\inf_{r \geqslant N} \frac{1}{r} \sum_{n-r < m < n} u_m + \sup_{r \geqslant N} \frac{v_{n-r}}{r} < 1\right) \\
&\leqslant P\left(\inf_{r \geqslant N} \frac{1}{r} \sum_{n-r \leqslant m < n} u_m < A\right) + P\left(\sup_{r \geqslant N} \frac{v_{n-r}}{r} < 1 - A\right)
\end{aligned}
$$

The last inequality holds for any $A \in \mathbb{R}$. Taking $A < Eu_n$, we get

$$
\lim_{N \to \infty} P\left(\inf_{r \geqslant N} \frac{1}{r} \sum_{n-r \leqslant m < n} u_m < A\right) = 0
$$

On the other hand, by assumption, $\lim P(w_n < N) = 1$, and hence,

$$
\lim_{N \to \infty} P\left(\sup_{r \geqslant N} \frac{1}{r} v_{n-r} < 1 - A\right) = 1
$$

Given $\varepsilon > 0$, take N so large that $P(\sup_{r \geqslant N} (1/r)\, v_{n-r} > 1 - A) < \varepsilon$. Then the bound

$$
\begin{aligned}
P\left(\sup_{r \geqslant 1} \frac{v_{n-r}}{r} > 1 - A\right) &\leqslant P\left(\sup_{r \leqslant N} \frac{v_{n-r}}{r} > 1 - A\right) \\
&\quad + P\left(\sup_{r \geqslant N} \frac{v_{n-r}}{r} > 1 - A\right) \qquad (2.7)
\end{aligned}
$$

implies that

$$
\lim_{K \to \infty} P\left(\sup_{r \geqslant 1} \frac{v_{n-r}}{r} > K\right) \leqslant \varepsilon
$$

Therefore, for any $\varepsilon > 0$,

$$
P\left(\sup_{r \geqslant 1} \frac{v_{n-r}}{r} < \infty\right) \geqslant 1 - \varepsilon
$$

and hence, a.e.

$$\sup_{r \geqslant 1} \frac{1}{r} v_{n-r} < \infty$$

By Lemma 2.1(ii), $Ev_n < \infty$. This proves the assertion 2. Theorem 5 is proved.

3. PROOF OF THE APPROXIMATION THEOREMS

The proofs of Theorems 2 and 3 are based on the same kind of arguments and, to avoid repetitions, only one of these statements (Theorem 3) will be proved. The modifications needed for proving Theorem 2 are immediate and are left to the reader. To start with, we also assume that the set J' contains just one element, a'_k, $k \in \mathbb{N}$.

We introduce the rmpp $\eta'_{k,t^0} = \eta'^{(N)}_{k,t^0}$, $1 \leqslant k \leqslant N$, $t^0 \in \mathbb{R}$, with marks $(L, W) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, $L = (l_0, l_1)$, $W = (w_0, w_1)$, which is obtained from the superposition of the rmpp's η_{j,t^0} $(= \eta^{(N)}_{j,t^0})$, $1 \leqslant j \leqslant N$, $t^0 \in \mathbb{R}^1$, by considering only the messages y with $d(y) = k$ and by omitting the value $d(y)$ from the mark. Taking into account that, due to the symmetry of our network, the distribution of the rmpp η'_{k,t^0} (as well as of the stationary rmpp η'_k) does not depend on k, we shall omit the index k whenever possible. The following assertion admits Theorem 3 as a Corollary, at least in the case $J' = \{k\}$.

Theorem 6. Assume that the bound (1.4) holds. Then for any $k \in \mathbb{N}$ and any bounded time interval $I \subset \mathbb{R}$, the following relation holds:

$$\lim_{t^0 \to -\infty, N \to \infty} \operatorname{Var}[\eta'^{(N)}_{t^0} 1_I, \eta'^{(\infty)} 1_I] = 0 \tag{3.1}$$

where $\eta'^{(\infty)}$ is the rmpp with marks (L, W) described in Theorem 3.

To prove Theorem 6, we perform an auxiliary construction which was used, under a slightly different form, in the proof of Theorem 5. Let ω be a realization of a rmpp with marks (L, W). Given $t \in \mathbb{R}$, we set

$$\begin{aligned} v_t(\omega) = \sup_{t' < t} \Bigg[& \sum_{y \in \omega, t' < t(y) < t} l_1(y) \\ & + w_0(y^F(t', t)) + l_0(y^F(t', t)) - w_0(y^L(t', t)) \\ & - l_0(y^L(t', t)) - (t - t') \Bigg]^+ \end{aligned} \tag{3.2}$$

where $y^F(t', t) = y^F(t', t, \omega)$ [respectively, $y^L(t', t) = y^L(t', t, \omega)$] is the first (resp. last) message of ω in the time interval (t', t) w.r.t. the order of epochs of arrival. Due to (2.6), one gets immediately that for any message $y \in \omega$, the following equality holds:

$$w_1(y) = w_0(y) + l_0(y) + v_{t(y)} \tag{3.3}$$

Let

$$z_t(\omega) = \sup[t' < t\colon v_{t'}(\omega) = 0] \tag{3.4}$$

and denote by ω_* the realization with marks (L, w_0), obtained from ω by deleting the component w_1. For any $t \in \mathbb{R}$, $u > 0$, and any pair of realizations ω and ω' such that (a) $\omega_* 1_{(t-u,t]} = \omega'_* 1_{(t-u,t]}$ and (b) $\min(z_t(\omega), z_t(\omega')) > t - u$, taking t_1 such that

$$\max(z_t(\omega), z_t(\omega')) < t_1 < t$$

we obtain that the following equality holds for every $\tilde{t} \in (t_1, t]$:

$$v_{\tilde{t}}(\omega) = v_{\tilde{t}}(\omega')$$

and hence

$$w_1(y, \omega) = w_1(y, \omega')$$

for every message $y \in \omega 1_{(t_1,t]} = \omega' 1_{(t_1,t]}$.

Roughly speaking, $z_t(\omega)$ is the "backward memory length" of the random variables $w_1(y, \omega)$. Hence, to prove Theorem 6, it is sufficient to check that:

I. For any $\varepsilon > 0$ there exists $u > 0$ such that for all $t^0 \in \mathbb{R}$, all $t \geqslant t^0$, and all $N \geqslant 1$,

$$P_{t^0}^{(N)}(z_t < t - u) < \varepsilon, \qquad P_{t^0}^{(\infty)}(z_t < t - u) < \varepsilon, \qquad P^{(\infty)}(z_t < t - u) < \varepsilon \tag{3.5}$$

where $P_{t^0}^{(N)}$ denotes the probability distribution of the rmpp $\eta'^{(N)}_{t^0}$, $P^{(\infty)}$ denotes that of the rmpp $\eta'^{(\infty)}$ defined by the properties (a) and (c) figuring in Theorem 3 and the property mentioned in the remark following Theorem 3, and $P_{t^0}^{(\infty)}$ is the distribution of the cutoff rmpp $\eta'^{(\infty)}_{t^0}$, which is obtained when replacing the reduced rmpp ξ by its restriction $\xi 1_{(t^0,\infty)}$.

II. For any $t \in \mathbb{R}$, any $t^0 \leqslant t$, and any $u > 0$,

$$\lim_{N \to \infty} \operatorname{Var}\{\zeta_{t^0}^{(N)} 1_{(t-u,t)}, \zeta_{t^0}^{(\infty)} 1_{(t-u,t)}\} = 0 \tag{3.6}$$

$$\lim_{N \to \infty} \operatorname{Var}\{\zeta^{(N)} 1_{(t-u,t)}, \zeta^{(\infty)} 1_{(t-u,t)}\} = 0 \tag{3.7}$$

where $\zeta_{t0}^{(\cdot)}$ and $\zeta^{(\cdot)}$ denote the rmpp's with marks $(L, w_0) \in \mathbb{R}_+^2 \times \mathbb{R}_+$ obtained from $\eta_{t0}'^{(\cdot)}$ and $\eta'^{(\cdot)}$ by deleting the component w_1.

We start with the proof of assertion I. From the definitions (3.2) and (3.3) it is easy to see that

$$\begin{aligned} &P(z_t < t-u) \\ &\quad < P\Bigg(\exists t' < t-u: \sum_{\tilde{y} \in \omega:\, t' < t(y) < t} l_1(y) + w_0(y^F(t', t)) \\ &\qquad + l_0(y^F(t', t)) + w_0(y^L(t', t)) + l_0(y^L(t', t)) - (t-t') > 0\Bigg) \end{aligned} \tag{3.8}$$

Here (and below) P stands for any of the probability distributions $P_{t_0}^{(N)}$, $P_{t_0}^{(\infty)}$, or $P^{(\infty)}$ figuring in (3.5).

One can check that for $u > 1$,

$$\begin{aligned} &\text{RHS of (3.8)} \\ &\quad \leqslant \sum_{n' \in \mathbb{Z}:\, n < t-u} P\Bigg(\exists t' \in [n, n+1): \\ &\qquad \times \sum_{y \in \omega:\, t' < t(y) < t} l_1(y) + w_0(y^F(t', t)) + l_0(y^F(t', t)) \\ &\qquad + w_0(y^L(t', t)) + l_0(y^L(t', t)) > t - t'\Bigg) \\ &\quad \leqslant \sum_{n:\, n < t-u} P\Bigg(\sum_{y \in \omega:\, n < t(y) < t} l_1(y) + w_0(y^F(n+1, t)) \\ &\qquad + l_0(y^F(n+1, t)) + \max_{y \in \omega:\, n \leqslant t(y) < n+1} [w_0(y) + l_0(y)] > t - n - 1\Bigg) \\ &\quad \leqslant \sum_{n:\, n < t-u} \Bigg[P\Bigg(\sum_{y \in \omega:\, n < t(y) < t} l_1(y) - M > \tfrac{1}{2}(1 + \lambda\mu^{-1})(t-n) - M - 1\Bigg) \\ &\qquad + P(w_0(y^F(n+1, t)) + l_0(y^F(n+1, t)) > \tfrac{1}{4}(1 - \lambda\mu^{-1})(t-n)) \\ &\qquad + P(\max_{y \in \omega:\, n \leqslant t(y) < n+1} [w_0(y) + l_0(y)] > \tfrac{1}{4}(1 - \lambda\mu^{-1})(t-n))\Bigg] \end{aligned} \tag{3.9}$$

where

$$M = E_P \sum_{y \in \omega:\, t' < t(y) < t} l_1(y)$$

We have used here condition (1.4) ensuring that $\frac{1}{4}(1-\lambda\mu^{-1})(t-n)>0$. Now notice that

$$M \leqslant \lambda\mu^{-1}(t-n)$$

(this is an equality for $P^{(\infty)}$ or for $P_{t^0}^{(N)}$ and $P_{t^0}^{(\infty)}$, provided that $n>t^0$), and hence

$$\tfrac{1}{2}(1+\lambda\mu^{-1})(t-n)-M \geqslant \tfrac{1}{2}(1-\lambda\mu^{-1})(t-n)$$

By using Chebyshev's inequality, we obtain that for n large enough,

$$\begin{aligned}\text{RHS of (3.8)} \leqslant C \sum_{n<t-u} \Bigg\{ & E_P\left(\left|\sum_{y\in\omega:\, n<t(y)<t} l_1(y)-M\right|^3\right)\frac{1}{(t-n)^3} \\ & + E_P[w_0(y^F(n+1,t))+l_0(y^F(n+1,t))]^2\frac{1}{(t-n)^2} \\ & + E_P[\max_{y\in\omega:\, n\leqslant t(y)<n+1}(w_0(y)+l_0(y))]^2\frac{1}{(t-n)^2}\Bigg\}\end{aligned} \tag{3.10}$$

where the constant C does not depend on t and N. Hence, it is sufficient to establish the following bounds, for $n<t-1$:

$$E_P\left(\left|\sum_{y\in\omega:\, n<t(y)<t} l_1(y)-M\right|^3\right)<C_1(t-n)^{3/2} \tag{3.11}$$

$$E_P[w_0(y^F(n+1,t))+l_0(y^F(n+1,t))]^2<C_2 \tag{3.12}$$

and

$$E_P[\max_{y\in\omega:\, n\leqslant t(y)<n+1}(w_0(y)+l_0(y))]^2<C_2 \tag{3.13}$$

where the constants C_1 and C_2 are independent of n, t, and N.

Finally, (3.11) is a particular case of a bound obtained first by Dharmadhikari and Jodgeo[10] (see also the book by Petrov,[17] Chapter III, Section 5, and the paper of Fook and Nagajev[12]). As to (3.12), this is nothing but the well-known bound for the $M/GI/1/\infty$ queue (see, e.g., Thorisson[18]). A similar meaning is extended to the bound (3.13).

As for the proof of II, we observe that the probability to have, in any of the rmpp's $\eta_{t^0}^{\prime(N)}$ and $\eta^{\prime(N)}$, on a bounded interval $(t-u, t)$, more than one message from the same peripheral source is $O(1/N)$ and hence tends to zero. But otherwise, the restrictions $\zeta_{t^0}^{\prime(N)}1_{(t-u,t)}$ and $\zeta_{t^0}^{\prime\infty}1_{(t-u,t)}$ coincide.

To prove Theorem 3 for the general case, it is sufficient to establish the following extension of Theorem 6.

Theorem 7. Under the condition (1.4), for any finite set $J' \subset \mathbb{N}$ and any bounded time interval $I \subset \mathbb{R}$,

$$\lim_{t^0 \to -\infty, N \to \infty} \mathrm{Var}[\{\eta'^{(N)}_{k,t^0} 1_I, k \in J'\}, \{\eta'^{(\infty)}_k 1_I, k \in J'\}^0] = 0$$

where $\{\eta'^{(\infty)}_k, k \in J'\}^0$ denotes the collection of independent copies of the rmpp $\eta'^{(\infty)}$ with marks (L, W) figuring in Theorems 3 and 6.

The key point in the proof of Theorem 7 is to check the bounds

$$P^{(N)}_{t^0}(z_{k,t^0} < t-u, k \in J') < \varepsilon \tag{3.14}$$

$$P^{(\infty)}_{t^0}(z_{k,t^0} < t-u, k \in J') < \varepsilon \tag{3.15}$$

$$P^{(\infty)}(z_{k,t^0} < t-u, k \in J') < \varepsilon \tag{3.16}$$

and the relations

$$\lim_{N \to \infty} \mathrm{Var}[\{\zeta^{(N)}_{k,t^0} 1_{(t-u,t)}, k \in J'\}, \{\zeta^{(\infty)}_{k,t^0} 1_{(t-u,t)}, k \in J'\}^0] = 0 \tag{3.17}$$

$$\lim_{N \to \infty} \mathrm{Var}[\{\zeta^{(N)}_k 1_{(t-u,t)}, k \in J'\}, \{\zeta^{(\infty)}_k 1_{(t-u,t)}, k \in J'\}^0] = 0 \tag{3.18}$$

where now $P^{(N)}_{t^0}$ denotes the joint probability distribution of the family of rmpp's $\eta'^{(N)}_{k,t^0}$, $k \in J'$, and $P^{(\infty)}$ and $P^{(\infty)}_{t^0}$ are the joint probability distributions of the (independent) limiting rmpp's $\eta'^{(\infty)}_k$, $k \in J'$, and $\eta'^{(\infty)}_{k,t^0}$, $k \in J'$, respectively. Correspondingly, $\{\zeta^{(\infty)}_{k,t^0}\}^0$ and $\{\zeta^{(\infty)}_k\}^0$ are collections of independent copies of the rmpp's $\{\zeta^{(\infty)}_{t^0}\}^0$ and $\{\zeta^{(\infty)}\}^0$ with marks (L, w_0). The proofs of these estimates are similar to those of (3.5) and (3.6)–(3.7), respectively, and are omitted for the sake of brevity.

4. STATIONARY SOLUTION OF THE GENERALIZED LINDLEY EQUATION AND THE ASSOCIATED BOUNDARY PROBLEM

The proof of Theorem 4 is immediate. Our probability density is of the form

$$g(x) = \frac{\lambda\mu}{\lambda+\mu} \{(\exp \lambda x)[1-\theta(x)] + [\exp(-\mu x)]\, \theta(x)\}, \qquad x \in \mathbb{R} \tag{4.1}$$

where θ is the indicator function of the nonnegative half-axis $\mathbb{R}_+$. For $x \neq 0$, we have

$$g''(x) + ag'(x) = \lambda\mu g(x) \tag{4.2}$$

and for $x=0$,

$$g'(0+)-g'(0-)=-\lambda\mu \tag{4.3}$$

Taking the first and the second derivatives of the equality (1.8), we arrive, in view of (4.2) and (4.3), at the assertions 1 and 2. For example,

$$\mathscr{F}''_{+}(x)=\int dr\,\mathscr{F}(r)\,\mathscr{F}_0(r)\,g''(x-r)-\lambda\mu\mathscr{F}(x)\,\mathscr{F}_0(x+0)$$

which is equivalent to (1.11). The other statements are deduced by inspection.

Remark. In the particular case where

$$\mathscr{F}_0(x)=1-\exp(-vx), \qquad x\geqslant 0$$

with $v>0$, Eqs. (1.10), (1.11) take the form

$$\mathscr{F}''(x)+a\mathscr{F}'(x)=\lambda\mu\exp(-vx)\,\mathscr{F}(x), \qquad x>0$$

which, after the change of variables

$$t=\frac{2(\lambda\mu)^{1/2}}{v}\exp\left(-v\frac{x}{2}\right)$$

$$\mathscr{F}=t^{\alpha}\mathscr{U}$$

where $\alpha=a/v$, becomes

$$\mathscr{U}''+\frac{1}{t}\mathscr{U}-\left(1+\frac{\alpha^2}{t^2}\right)\mathscr{U}=0 \tag{4.4}$$

Equation (4.4) is a Bessel-type equation. Its general solution is given by a linear combination

$$\mathscr{U}(t)=c_1 I_\alpha(t)+c_2 K_\alpha(t)$$

where I_α is the modified Bessel function and K_α is the Macdonald function, both of order α. The constants c_1, c_2 may be found from the constraints (1.9). A similar formula was obtained by Baccelli *et al.*[2]

A solution of the stochastic equation (1.6) [or, equivalently, a non-decreasing solution $\mathscr{F}$ of the functional equation (1.8) with the constraints (1.9)] may be interpreted as a stationary solution of the generalized Lindley equation (2.1). It seems interesting to investigate the connection between solutions to (1.8) and solutions to (1.10)–(1.13) (we shall call the last problem the associated boundary problem). To make the idea more transparent, we consider a more general situation. Let $\mathscr{L}$ be a differential operator with constant coefficients of order $n > 2$:

$$\mathscr{L} = \frac{d^n}{dx^n} + b_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + b_1\frac{d}{dx} + b_0, \qquad x \in \mathbb{R} \tag{4.5}$$

and let h_1, h_2 be two solutions of the equation $\mathscr{L}h = 0$ such that

$$h_1(0) = h_2(0), \qquad h_1'(0) = h_2'(0),..., h_1^{(n-2)}(0) = h_2^{(n-2)}(0) \tag{4.6}$$

and

$$h_1^{(n-1)}(0) - h_2^{(n-1)}(0) = c \tag{4.7}$$

We set

$$g = h_1\theta + h_2(1-\theta) \tag{4.8}$$

and consider the functional equation (1.8) supposing that $\mathscr{F}$ is a locally bounded measurable function. Denote by $\mathscr{A}$ $[=\mathscr{A}(n, g, \mathscr{F}_0)]$ the class of measurable functions $\mathscr{F}$ such that:

(i) $\mathscr{F}\mathscr{F}_0$ is locally bounded.

(ii) For any $k = 0,..., n$, the integral $g^{(k)} * (\mathscr{F}\mathscr{F}_0)$ converges.

Lemma 4.1. Let $\mathscr{F} \in \mathscr{A}$ be a solution of (1.8). Then:

1. $\mathscr{F}$ is of class C^{n-1}.
2. If $\mathscr{F}_0$ is of class C^0 (continuous), then $\mathscr{F}$ is of class C^n.
3. For any $x \in \mathbb{R}$,

$$\mathscr{L}_{+,-}\mathscr{F}(x) - c\mathscr{F}(x)\,\mathscr{F}_0(x+, -0) = 0 \tag{4.9}$$

where $\mathscr{L}_{+,-}$ denotes the left-hand (respectively, right-hand) version of $\mathscr{L}$.

Conversely, let $\mathscr{F}_0$ be continuous and let $\mathscr{B}$ $[=\mathscr{B}(n, g)]$ denote the class of functions $\mathscr{F} \in C^n$ such that:

(i) The integrals $g * \mathscr{F}^{(k)}$ converge, $k = 0,..., n$.

(ii) $g * \mathscr{F}^{(k)} = g^{(k)} * \mathscr{F}$, $k = 0,..., n$.

Lemma 4.2. Assume that the constant c in (4.7) is nonzero. Then any $\mathscr{F} \in \mathscr{B}$ satisfying (4.9) gives a solution to (1.8).

The proof of both Lemmas 4.1 and 4.2 is an immediate extension of that of Theorem 4 [the case of Theorem 4 corresponds to $\mathscr{L}$ given by the RHS of (4.2)]. Notice that the equation $\mathscr{L}h = 0$ has solutions h_1, h_2 such that the function (4.8) is nonnegative and integrable iff the characteristic polynomial $\hat{\mathscr{L}}(s)$ has at least one root on the negative half-axis and at least one root on the positive half-axis.

Concluding this section, we formulate some problems which are of interest in this context. Suppose an operator $\mathscr{L}$ of the form (4.5) is given, and h_1, h_2 are solutions of $\mathscr{L}h = 0$ such that the function (4.8) is nonnegative and integrable. Let $\mathscr{F}_0$ be nonnegative, nondecreasing, and locally bounded (or even globally bounded).

1. When does Eq. (1.8) have nondecreasing and/or nonnegative solutions?

2. When does Eq. (1.8) have bounded solutions?

3. When does Eq. (1.8) have a unique solution (up to the multiplicative constant)?

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